Entropy: from black holes to ordinary systems

2006 J. Phys. A: Math. Gen. 39 7175

(http://iopscience.iop.org/0305-4470/39/23/001)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **39** (2006) 7175–7185 doi:10.1088/0305-4470/39/23/001

Entropy: from black holes to ordinary systems

J P Badiali

LECA, ENSCP-Université Pierre et Marie Curie, 4 Place Jussieu, 75230 Paris Cedex 05, France

E-mail: badiali@ccr.jussieu.fr

Received 7 February 2006, in final form 30 March 2006
Published 23 May 2006
Online at stacks.iop.org/JPhysA/39/7175

Abstract

Several results of black holes thermodynamics can be considered as firmly founded and formulated in a very general manner. From this starting point, we analyse in which way these results may improve our understanding in the thermodynamics of ordinary systems for which a pre-relativistic description is sufficient. First, we introduce a spacetime model and an entropy related to a local definition of the order in this spacetime. We show that such an approach leads to the traditional thermodynamics provided an equilibrium condition is assumed. From this condition a relation time/temperature is introduced. We show that such a relation extensively used in the black hole theory has a very general and physical meaning here. Our dynamical approach of thermodynamic equilibrium allows us to establish a relation between action and entropy identical to the one existing in the case of black holes. Since this relation exists for systems with very different underlying physics, we may expect that it corresponds to a general result in thermodynamics; it suggests that a definition of entropy in terms of order in spacetime might be more general than the Boltzmann definition related to a counting of microstates. All these results based on the fact that the paths introduced in the path-integral formalism have a physical meaning give a new approach of statistical mechanics. Finally, we compare our approach to other works based on a similar starting point.

PACS numbers: 03.65.Ca, 05.30.−d, 05.70.−a, 47.53.+n

1. Introduction

After the discovery of the Hawking radiation [1], it became clear that the similarity observed by Bekenstein [2] between the four laws of black hole mechanics and the ones of standard thermodynamics is much more than a simple analogy but represents a result having a deep significance. For a given black hole, the same values for temperature, T_{BH}, and entropy, S_{BH}, have been obtained from different approaches. Consequently, we may consider the black hole thermodynamics as firmly established; one exception, may be, concerns the derivation of the second law of thermodynamics that is replaced by the so-called generalized second law [3]. The main question emerging from these results is relative to the physical origin of the black hole entropy. It is implicitly assumed that the physical content of S_{BH} can be understood, at least to some extent, from the usual ingredients of classical physics. This leads to search a relation between S_{BH} and a counting of microstates. After 30 years of intensive works in that direction, we have still to deal with questions relative to the nature and the location in space of these microstates [4, 5].

In this paper we reverse this problematic. We do not try to understand something concerning the black hole physics starting from the results obtained in ordinary physics but we try to see if it is possible to learn something about ordinary systems starting from results obtained in the domain of black holes. By ordinary systems, we mean those for which a pre-relativistic description is sufficient. Obviously, we cannot expect a strict mapping between the thermodynamic properties of ordinary systems and those of systems including black holes for which there are fundamental singularities, the presence of horizons and the existence of a holographic principle. Nevertheless, as we shall see, it is possible to reformulate some results obtained in the black hole domain in a so general manner that it appears natural to ask whether there exists similar results in the case of ordinary systems.

It is well known that the Hawking temperature, T_H, of a Schwarzschild black hole of mass M measured at a large distance from the hole is given by $T_H = \frac{\hbar}{8\pi M}$. In the original derivation of this result, Hawking [1] used techniques of quantum field theory on a given classical curved background spacetime. For this purpose, it was useful to work in the 'real Euclidean section' of the Schwarzschild geometry, in which the time is rotated to its imaginary value and the thermal Green function is periodic with a period $\beta\hbar = \frac{\hbar}{k_B T_H}$ (see for instance [6] for a mathematical analysis of this result). Later, Gibbons and Hawking [7] were able to deduce thermodynamics of black hole from statistical mechanics. The main steps in their derivation are the following. First, they started from the existence of a partition function, Z_{GH}, defined according to

$$Z_{\mathrm{GH}} = \int \mathcal{D}[g] \exp -\frac{1}{\hbar} A^E[g] \tag{1}$$

in which $A^E[g]$ is the Euclidean action of the gravitational field associated with the metric g and $\mathcal{D}[g]$ means that we have to perform a functional integration over all the possible metrics. In (1), the action is defined on a time interval $\beta\hbar = \frac{\hbar}{k_B T_H}$. Then, they performed a zero loop approximation from which the free energy defined by $F_{\mathrm{GH}} = -k_B T_H \ln Z_{\mathrm{GH}}$ is directly proportional to the action. Finally, the entropy was deduced from F_{GH} by using the usual thermodynamic relations. For four different metrics, it has been shown that the results derived in this way agree with previous derivations although the methods are totally different. Thus, the entropy derived from the spacetime properties has a geometrical origin connected with the spacetime metrics, it is not based explicitly on the counting of microstates.

It is tempting to retain from these results some general aspects that we may try to extend to ordinary systems. First, in parallel to the usual definition of entropy it might exist an alternative definition connected with the geometry of spacetime and that we might establish without explicit reference with a counting of microstates. In what follows, we shall see that such an alternative definition of entropy also exists for ordinary systems provided a spacetime model is introduced [8]. Second, black hole theory suggests the existence of a relation between action and spacetime free energy leading to a relation between action and entropy. Hereafter, we deduce an identical relation valid for ordinary systems. For the simple case considered here, the result is quite general, i.e., it is not restricted to the zero loop approximation. It is noteworthy that the existence of a relation between action and entropy has been suggested by

Eddington a long time ago and reinvestigated later by de Broglie searching a relation between two quantities that are considered as relativistic invariants in restricted relativity (for a review in this domain see [9]).

This paper is organized as follows. In section 2, we introduce a spacetime model similar to the one presented in [8]. In section 3, we define the path entropy. In section 4, we analyse the relation time/temperature from which we establish a link between our spacetime approach and ordinary thermodynamics; a new heuristic approach of this relation is given. In section 5, we derive the relation between action and entropy. In section 6, we compare some points of our approach with recent works starting from a similar basis. In the last section we give some concluding remarks.

2. Spacetime model

In his book with Hibbs, Feynman [10] developed a given number of fundamental remarks concerning the derivation of the partition function in terms of path integral. He suggested a possible new foundation of statistical mechanics directly in terms of path integral as he did for the amplitude of probability in quantum mechanics; in this context, 'directly' means without using the Schrödinger equation. This Feynman's conjecture implies to give a physical meaning to the paths. Excepted in a few number of attempts (see section discussion), the paths are considered as a mathematical trick without any connection with a real motion in spacetime. To give a physical meaning to the paths, we may reconsider our knowledge about the structure of spacetime.

Today there are many indications showing that spacetime may be discrete rather continuous (see for instance [11] and the references quoted therein). In the light of recent works, it is clear that the discretization of spacetime appears as natural when a minimal length exists. However, this discretization is much more than the reduction of usual continuous equations to their discrete forms, it leads to a drastic change in our description of the microscopic world. This can be illustrated starting from the seminal paper of Snyder [12] in which it has been shown that the presence of the Compton wavelength, λ_C, does not destroy the Lorentz invariance of spacetime provided we change the standard commutations rules of quantum mechanics. Now, the position operators relative to two different orientations do not commute anymore; this leads to introduce a non-commutative geometry from which it is possible to immediately recover the Dirac equation [13]. The spectrum of position operators consists of values such as $m\lambda_C$ where m is an integer. λ_C represents the best accuracy with which we can localize a particle: for distances smaller than λ_C a multiparticle theory is needed.

A similar situation appears in the domain of quantum gravitation where it has been shown that for distances shorter than the Planck length, λ_P, the concept of spacetime loses its meaning [14]. New commutations rules are established using string theory or a heuristic combination of quantum mechanics and standard general relativity (see for instance [15]). From the new commutations rules, it is possible to establish an algebra for operators also leading to a non-commutative geometry. Recent papers analyse the algebraic structures related to this generalized uncertainty principle.

Another approach developed in relation with quantum gravity is the causal set theory in which we combine spacetime discreetness and causality (for an introduction in this domain see [16, 17]). Here, causality appears as a fundamental organizing principle, spacetime is replaced by an assembly of discrete elements organized by means of relations between them into a partially ordered set. To build up a causal set theory, it is natural to assume that the discrete spacetime has a deep structure containing some ingredients that are already familiar to us from our study of the world at a larger scale. This represents a new kind of correspondence principle between a discrete spacetime and its continuous limit. Quite naturally causal set theory leads to consider random walks and to wonder about the existence of a quantum stochastic process that should lead to the Schrödinger equation in a continuous limit [18].

For the problem that we have in mind, i.e., to establish a relation between action and entropy in the case of ordinary (pre-relativistic) systems a simple model of spacetime can be used, as we shall see the one introduced in [8] is sufficient. Hereafter, we will briefly summarize the main points of this model. We assume that the spacetime points (t_i, x_i) are located on the sites of a regular lattice as in the chessboard model investigated in [10]. The spacetime structure is characterized by the existence of a relation between the elementary length Δx and time interval Δt corresponding to the lattice spacing. We assume that the free motion in this spacetime is as simple as possible. By definition, a path corresponds to a set of sites (t_i, x_i); the values of t_i are such as $t_{i+1} > t_i$ whatever i and the coordinate positions, x_{i+1} is necessarily one of the nearest neighbours of x_i, thus a path corresponds to a random walk.

In this spacetime model we have a discrete manifold, the quantification appears via the relation between Δx and Δt and the kinematics is defined in terms of paths on which a causal relation exists. In this model there is no metric but a causal structure as in the theory of causal sets [16, 17]. When a mass is introduced in this lattice, we assume that $(\Delta x)^2/\Delta t = \hbar/m$, a relation mimicking the Heisenberg uncertainty relations [8]. Note that this relation does not fix the values of Δx and Δt. In the absence of gravity and if we assume that the velocity of light is infinite both the Planck and the Compton lengths vanish, then there is no natural unit of length. Therefore, we may extend the previous relation in the limits $\Delta x, \Delta t \to 0$. The initial discreetness persists because these limits are taken with the constraint $(\Delta x)^2/\Delta t = \hbar/m$. This leads to a continuous diffusion process [19] for which the diffusion coefficient is $D = \hbar/2m$. For this diffusion process we note $q_0(t_0, x_0; t, x)$ the density of transition probability to go from (t_0, x_0) to (t, x) when $t \geqslant t_0$. From $q_0(t_0, x_0; t, x)$ and a function $\phi_0(x)$ defined for $t = t_0$, we form the function $\phi(t, x)$ according to

$$\phi(t, x) = \int \phi_0(y) q_0(t_0, y; t, x)\, \mathrm{d}y \tag{2}$$

which is the solution of the diffusion equation

$$-\partial\phi(t, x)/\partial t + D\Delta_x \phi(t, x) = 0 \tag{3}$$

verifying the initial-value problem $\phi(t_0, x) = \phi_0(x)$. Note that $q_0(t_0, x_0; t, x)$ is the fundamental solution of (3) in which Δ_x is the Laplacian operator taken at the point x.

In the presence of an external potential, $u(t, x)$, we generalize (3) into

$$-\partial\phi(t, x)/\partial t + D\Delta\phi(t, x) - \frac{1}{\hbar} u(t, x)\phi(t, x) = 0. \tag{4}$$

In contrast to (3), the fundamental solution of (4), $q(t_0, x_0; t, x)$, cannot be normalized in general [20]. Thus, $q(t_0, x_0; t, x)$ is no more a transition probability density but it verifies the Chapman–Kolmogorov law of composition [20] and therefore it can be used to describe transitions in spacetime. Using the Feynman–Kac formula, the fundamental solution, $q(t_0, x_0; t, x)$, of this new equation appears as a weighted sum of all the paths connecting the spacetime points (x_0, t_0) to (x, t); we have

$$q(t_0, x_0; t, x) = \int \mathcal{D}x(t) \exp -\frac{1}{\hbar} A^E[x(t); t, t_0] \tag{5}$$

where $\mathcal{D}x(t)$ means the measure for the functional integral and

$$A^E[x(t); t, t_0] = \int_{t_0}^{t} \left[\frac{1}{2} m \left[\frac{\mathrm{d}x(t')}{\mathrm{d}t'} \right]^2 + u(t', x(t')) \right] \mathrm{d}t'. \tag{6}$$

From (5), (6) and a given function $\phi_0(x)$, we may construct the solution of (4) verifying the initial condition $\phi(t_0, x) = \phi_0(x)$ by changing $q_0(t_0, x_0; t, x)$ into $q(t_0, x_0; t, x)$ in (2).

At this level, it is important to underline that the paths are associated with processes that occur in real time. These processes are generated from the Euclidean action $A^E[x(t); t, t_0]$ and they are such that, in average, there is no derivative, i.e., no velocity in the usual sense on the paths [21]. In what follows, we will show that the function $q(t_0, x_0; t, x)$ is sufficient to give a description of the order in spacetime.

3. Path entropy

To define the order—or disorder—in spacetime, we first adopt a local definition. Around a point x_0, we count the number of closed paths that we can form during a given time interval τ. If there is only one possible path we can say that we have a perfect order, no fluctuation around this path is accepted. However, after introducing a given measure, some fluctuations can take place and we have to deal with a given number of acceptable paths. For this measure, we associate the order in spacetime with this number of paths. The total order in our system will be obtained by summing the result of this procedure on all the points x_i. This definition seems quite natural anytime we have to deal with processes occurring in a given spacetime. Of course, such a definition is not unique but it is probably the simplest one.

By analogy with the thermodynamic entropy, which is defined for given values of internal energy and volume, we consider that our spacetime system is prepared with a given energy U and filled with a volume V. We define a path entropy by counting the number of paths for which the Euclidean action that we note hereafter as $A^E[x(t); \tau]$ does not deviate too much from the action τU. In reference with the standard thermodynamics, we define a path entropy, S_{path}, according to

$$S_{\mathrm{path}} = k_B \ln \int \mathrm{d}x_0 \int \mathcal{D}x(t) \exp - \frac{1}{\hbar}[A^E[x(t); \tau] - \tau U]. \tag{7}$$

S_{path} can be also rewritten as

$$S_{\mathrm{path}} = \frac{k_B \tau}{\hbar} U + k_B \ln Z_{\mathrm{path}} \tag{8}$$

with

$$Z_{\mathrm{path}} = \int \mathrm{d}x_0 \int \mathcal{D}x(t) \exp - \frac{1}{\hbar} A^E[x(t); \tau] = \int \mathrm{d}x_0\, q(0, x_0; \tau, x_0) \tag{9}$$

in which $q(0, x_0; \tau, x_0)$ corresponds to closed paths observed during a time interval τ and for which $t_0 = 0$. Z_{path} is the total number of closed paths that we may count during τ irrespective the value of U. S_{path} contains two external parameters τ and U while Z_{path} is only function of τ. We may characterize the dependence of S_{path} versus these two parameters by considering the two derivatives: $\frac{\mathrm{d}S_{\mathrm{path}}}{\mathrm{d}U}$ defined as $\frac{1}{T_{\mathrm{path}}}$ and $\frac{\mathrm{d}S_{\mathrm{path}}}{\mathrm{d}\tau}$. From the results given in [8], we have

$$\frac{\hbar}{k_B} \frac{\mathrm{d}S_{\mathrm{path}}}{\mathrm{d}U} = \frac{\hbar}{k_B} \frac{1}{T_{\mathrm{path}}} = \tau + [U - (\langle u_K \rangle_{\mathrm{path}} + \langle u_P \rangle_{\mathrm{path}})] \frac{\mathrm{d}\tau}{\mathrm{d}U} \tag{10}$$

and

$$\frac{\hbar}{k_B} \frac{\mathrm{d}S_{\mathrm{path}}}{\mathrm{d}\tau} = [U - (\langle u_K \rangle_{\mathrm{path}} + \langle u_P \rangle_{\mathrm{path}})] + \tau \frac{\mathrm{d}U}{\mathrm{d}\tau} \tag{11}$$

in which a relation between U and τ is assumed. The averages over paths that appear in (10) and (11) are defined according to

$$\langle u_P \rangle_{\mathrm{path}} = \frac{1}{Z_{\mathrm{path}}} \int \mathrm{d}x_0\, u(x_0) q(0, x_0; \tau, x_0) \tag{12}$$

and

$$\frac{m}{2}\left\langle\left(\frac{\delta x}{\delta t}\right)^2\right\rangle_{\text{path}} = \frac{\hbar}{2\delta t} - \langle u_K\rangle_{\text{path}} \tag{13}$$

in which $\langle u_K\rangle_{\text{path}}$ is a well-behaved function in the limit $\delta t \to 0$; moreover we have checked on examples that $\langle u_K\rangle_{\text{path}}$ is just the usual thermal kinetic energy [8].

The quantities, $S_{\text{path}}, Z_{\text{path}}, T_{\text{path}}$ and $\frac{\mathrm{d}S_{\text{path}}}{\mathrm{d}\tau}$ are well-defined, they gives us a global characteristic of the spacetime structure but none of them corresponds to a thermodynamic quantity. In the next section, we show that there is a value of τ for which a correspondence can be established between these quantities and thermodynamic properties.

4. Relation between time and temperature

In (10), the sum $\langle u_K\rangle_{\text{path}} + \langle u_P\rangle_{\text{path}}$ is a well-defined quantity depending on the parameter τ and we may choose a particular value of τ in such a way that the previous sum coincides with U. This choice corresponds to a condition of thermal equilibrium, the energy that we spent on the paths corresponds to the energy of system formation, i.e., U. Now from (10) we conclude that the relation between τ and the temperature T_{path} is $\tau = \frac{\hbar}{k_B T_{\text{path}}}$ whatever the value of $\frac{\mathrm{d}\tau}{\mathrm{d}U}$ and whatever the potential $u(x)$. If we identify T_{path} with the usual temperature, we can see that Z_{path} defined in (9) becomes

$$Z = \int \mathrm{d}x_0 \int \mathcal{D}x(t) \exp - \frac{1}{\hbar}\int_0^{\beta\hbar}\left[\frac{1}{2}m\left[\frac{\mathrm{d}x(t')}{\mathrm{d}t'}\right]^2 + u(x(t'))\right]\mathrm{d}t' \tag{14}$$

which is identical to the traditional partition function expressed in terms of path integral [10] and thus we may recover all the results of thermodynamics. It is quite simple to verify that (11) is now reduced to $\frac{\hbar}{k_B}\frac{\mathrm{d}S_{\text{path}}}{\mathrm{d}\tau} = \tau\frac{\mathrm{d}U}{\mathrm{d}\tau}$ that we can rewrite as $\mathrm{d}U = T\,\mathrm{d}S$. Here, we may interpret this relation as follows: if we increase the energy U for the system preparation we increase the number of paths available and therefore the entropy in spacetime. It can be shown that τ also corresponds to the time interval that we have to wait in order to relax the quantum fluctuations and to reach a thermal regime [22].

Finally, we may also derive the relation $\tau = \beta\hbar$ from the following heuristic argument. From standard thermodynamics we know that if a change of energy ΔU produces on a moving body a change of momentum ΔP, the corresponding change of entropy ΔS is given by [23]

$$\Delta S = \frac{1}{T}\Delta U - \left(\frac{V}{T}\right)\Delta P \tag{15}$$

in which V is the velocity of the mobile. From (15) we may learn two different kinds of results. First, in terms of variations we may assume that ΔU results from quantum fluctuations and its estimation is $\Delta U = \frac{\hbar}{2\tau}$ while the dynamic part can be written as $\Delta\left(\frac{1}{2}mV^2\right)$ and if we assume that the quantum fluctuations lead to the thermal equilibrium we have $\Delta\left(\frac{1}{2}mV^2\right) = \frac{1}{2\beta}$. Since the system is at equilibrium, we must have no net change of entropy during these fluctuations; we can see that $\Delta S = 0$ implies $\tau = \beta\hbar$. Second, relation (15) is adequate for an interpretation in the domain of restricted relativity. Since the entropy is considered as a relativistic invariant, the right-hand side of equation (15) must be relativistic invariant [23]. It appears as the scalar product of the energy momentum tensor by a quantity $\left(\frac{1}{T}, \left(\frac{V}{T}\right)\right)$ that must be a 4-vector showing that $\left(\frac{1}{T}\right)$ must behave as a time in a Lorentz transformation. This result, added to others developed in [8], supports the idea that it exists a fundamental relation between time and the reverse of temperature.

The relation $\tau = \beta\hbar$ is extensively used in the derivation of the black hole properties but it appears more as a mathematical trick than a relation having a strong physical basis. A simple way to justify the value of τ has been proposed by Hawking starting from the analogy between the calculation of (1) and the usual method of quantum field theory [24]. Since the Euclidean action for the gravitational field and the thermal Green functions are periodic in imaginary time [6], it seems logical to identify the two periods. This leads to introduce a time and its relation with the temperature. What is behind this identification is not totally clear nevertheless τ is considered as a real time. This is quite clear in a paper of Padmanabhan [25] devoted to systems with a horizon. In this case, the gravitational action is calculated in real time, then the value of τ appears as an extra hypothesis.

In our derivation of the relation $\tau = \beta\hbar$, we do not use the Schrödinger equation or the canonical form of the density matrix that is needed to find the properties of the Green function. We consider an equilibrium condition and what we need from quantum mechanics is the existence of a relation that mimics the Heisenberg uncertainty relation. This is reminiscent of a strong result obtained by Wald [3] showing that it is possible to derive some properties of black hole without using the detailed expression of the Einstein equation.

From all arguments developed above, it seems normal to conclude that the relation $\tau = \beta\hbar$ has a very general and deep physical meaning.

5. Relation between action and entropy

In the Gibbs ensemble approach of statistical mechanics, we cannot expect a relation between action and entropy because these two quantities have a different nature: entropy is considered as an equilibrium quantity that we have to calculate by integration over the phase space while the action is a dynamical quantity and its definition requires the introduction of a time interval. In the previous section, we have developed a dynamical approach of thermodynamic equilibrium in which a time interval τ is associated with the reverse of the temperature. We have shown that the free energy defined according to $F = -k_B T \ln Z$ is a functional of an Euclidean action $A^E[x(t); t, t_0]$. Before setting up a relation between action and entropy, we analyse the relation between the Lagrangian and Euclidean version of the action.

In (12) we have defined a potential energy $\langle u_P \rangle_{\text{path}}$ by an average over paths and the mean value of the kinetic energy over paths will be defined by $\frac{m}{2}\left\langle\left(\frac{\delta x}{\delta t}\right)^2\right\rangle_{\text{path}}$ that we have introduced in (13). As usually we may define a Lagrangian by the difference between kinetic and potential energies. For a given temperature T, the average over the paths of this Lagrangian $\langle L(T)\rangle$ is given by

$$\langle L(T)\rangle = \left[\frac{m}{2}\left\langle\left(\frac{\delta x}{\delta t}\right)^2\right\rangle_{\text{path}} - \langle u_P\rangle_{\text{path}}\right] = \frac{\hbar}{2\delta t} - [\langle u_K\rangle_{\text{path}} + \langle u_P\rangle_{\text{path}}] = \frac{\hbar}{2\delta t} - U. \qquad (16)$$

The second equality results from (13) and the third is the consequence of the equilibrium condition discussed in the previous section. Thus, from (16) we can see that $\langle L(T)\rangle$ is ill defined if the time interval δt on which we calculate the kinetic energy goes to zero. Instead of $\langle L(T)\rangle$, we introduce the product

$$\langle A(T, \delta t)\rangle = \langle L(T)\rangle\delta t = \frac{\hbar}{2} - U\delta t \qquad (17)$$

that is a well-defined quantity whatever the value of δt. We may consider $\langle A(T, \delta t)\rangle$ as the elementary Lagrangian action over the paths. In the limit $\delta t \to 0$, we see that $\langle A(T, \delta t)\rangle$ corresponds to the quantum of action. In (17), $U\delta t$ represents an Euclidean action since it contains the sum of the kinetic and potential energies. Thus, (17) gives the relation between the Lagrangian and the Euclidean actions in our spacetime model. For a fixed value of δt, these two actions vary with opposite sign. If we increase the temperature by δT, we have

$$\langle A(T+\delta T, \delta t)\rangle - \langle A(T, \delta t)\rangle = -\delta t[U(T+\delta T) - U(T)] = -\delta t T\delta S \qquad (18)$$

in which the last equality is a consequence of the usual thermodynamic relation. The net change of action δA on the time interval τ will be obtained by summing up $[\langle A(T+\delta T, \delta t)\rangle - \langle A(T, \delta t)\rangle]$ on all the elementary time interval covering the total time interval τ. In the last part of (18), this will be simply done by multiplying the previous result by the number of elementary steps, i.e., $\frac{\tau}{\delta t}$. It is easy to see that the final result can be written as

$$\frac{\delta A}{\hbar} = -\frac{\delta S}{k_B}. \qquad (19)$$

This is an important result of this paper. It has been established for ordinary systems and shows that a change of entropy is equivalent to a change in the mean value of action calculated over the paths. Note that the existence of such a relation has been suspected by de Broglie [9] using relativistic arguments. For a black hole having an area $\mathcal{A}$, the entropy is given by $S = \frac{k_B c^3}{4G\hbar}\mathcal{A}$ and the Euclidean action is $A^E = -\frac{c^3}{4G}\mathcal{A}$ leading to the relation $\frac{S}{k_B} = -\frac{A^E}{\hbar}$ from which we immediately get (19) provided we identify the change δA with δA^E. This identification is justified by the fact that the Euclidean action in classical mechanics and in quantum field theory is defined with an opposite sign (see for instance [26]). It is important to underline that (19) results from a zero loop approximation in the case of black hole while we have an exact derivation here. It is also interesting to note that (19) is verified for systems with very different underlying physics; in the case of ordinary systems, the Euclidean action is given by (6) while in the case of black hole the action is the gravitational one. This fits quite well with the spirit of thermodynamics and we may think that (19) represents a very general result.

6. Discussion

Entropy is a fundamental quantity in physics and the law of evolution of entropy may be considered as the actual law of system evolution [27]. Recently, assuming the proportionality between entropy and horizon area for all local accelerated horizons, Jacobson [28] derived the Einstein equations as an equilibrium equation of state. For these reasons, it seems interesting to focus first on statistical mechanics considered as a good representation of thermodynamics and later to derive a Schrödinger equation as we have done in [8]. This leads to reverse the traditional route beginning with the Schrödinger equation and ending with the expression of the partition function via the use of the canonical form for the density matrix. The route we follow is also in the spirit of standard thermodynamics in which it is claimed that physical processes are basically irreversible but that we may create some reversible processes by imposing a symmetry between initial and final states [29]. This approach forces us to carefully analyse the problem of time irreversibility. In statistical mechanics, the motion over the paths is characterized by a positive semi-group from which we have derived a like H-theorem [8]. However, if we force the system to have a time-reversible behaviour then, as shown in [8], we may recover a Schrödinger equation using some mathematical results established by Nagasawa [20, 30]. Now, instead of a positive semi-group we have to deal with an unitary group.

It is important to note that our work is based on the diffusion equation in an external field (4) that it is not a Fokker–Planck equation and its solution is not a density of probability. Thus, our definition of entropy is not related to the existence of a Markovian process.

In order to derive a Schrödinger equation from this kind of approach, we have to consider two diffusion equations like (4) *in duality* relative to a given measure as shown in [20]. Then, in addition to $\phi(t, x)$ we have to consider a second function $\hat{\phi}(t, x)$ and two times t_0 and t_1 for which two boundaries conditions are fixed *independently* for $\phi(t, x)$ and $\hat{\phi}(t, x)$. The Schrödinger equation works for $t_0 \geqslant t \geqslant t_1$. Clearly, in statistical physics or in quantum mechanics we have not to deal with the simple Markovian process analysed in [31] and the results obtained in [31] or in [32] are irrelevant here. Our approach is not based on the so-called stochastic mechanics. If for practical reasons we know the density of probability $\phi(t, x)\hat{\phi}(t, x)$ at a time t_0 *or* t_1, it may be difficult to know the functions $\phi(t_0, x)$ and $\phi(t_1, x)$ and to describe the physics in terms of these functions. Fortunately, in this practical case we may use a reverse theorem established by Nagasawa [20] showing that the Schrödinger equation implies the existence of two diffusion equations in duality. By using this theorem, we may determine for instance what kind of potential $u(t, x)$ in spacetime may generate a Gaussian wavepacket for free particles [33].

In the continuous limit, our primarily discrete spacetime leads to (3) in the case of free particles therefore it should be possible to start our derivation of statistical mechanics directly from this diffusion equation. In our approach, we do not explain why the diffusion plays a preeminent role but we show that this equation may appear from general schemes used in other domains of quantum physics as those exposed in section 2. From them we can see that a natural extension of our work should consist in introducing new uncertainty relations related to the new commutations rules as those introduced in the paper of Snyder [12] and leading to a non-commutative geometry.

We may also note that our spacetime model is well defined since we give a clear answer to the following dilemma [34]: is the geometry of the underlying spacetime fractal or is the underlying spacetime regular and the fractal character generated by the dynamics? Here, we show that for the problems under consideration the second assumption is sufficient. There is no need to introduce a fractal spacetime as suggested in [35, 36]. However, if there is conceptual differences between our work and those based on a fractal spacetime, there also exists some similarities. This can be illustrated by considering a recent publication of Nottale *et al* [37]. These authors do not use a Fokker–Planck equation and their results, as ours, are not related to stochastic mechanics. The existence of two values for the velocity is a key concept in [37] and in [8] it was crucial to understand a H-theorem. An interesting aspect of their work is a discussion about the transition between quantum and classical physics in their scale relativity. This transition is governed by a characteristic time that they identify with the Einstein–de Broglie scale of the system, i.e., $\frac{\hbar}{E}$ in which E is a characteristic energy. In our approach if we associate E with the thermal energy we find that this characteristic time is τ, a meaningful result since we have seen that τ is the time interval separating a quantum domain from a thermodynamical one. Also note that it cannot be totally excluded that a connection between the non-commutative geometry mentioned above and some kind of fractality might exist [38].

Our work is based on two points. First, we want to describe statistical mechanics and then quantum physics at the level of the Schrödinger equation; this order leads to introduce a simple model of spacetime and kinematics. Second, we give a physical meaning to the paths. This is not a new idea and there is several attempts in which the paths are considered as physical entities. In this domain, the work of Nelson [39] represents an important step; in this approach the diffusion coefficient is the same as the one used in this paper. However, as mentioned above our work is not based on stochastic mechanics. Ancient works in which a physical meaning has been associated with the paths have been summarized in [30]. Very recent and new attempts in this field are due to Ord (see for instance [40, 41]) who tries to recover the properties of the wavefunction using only ensembles of classical point particles moving on continuous trajectories in spacetime. From this point of view, it was possible to derive the Schrödinger and the Dirac equations. In the papers of Ord, the role of time irreversibility is carefully analysed and the reversibility of the Schrödinger equation is obtained by considering more information on the paths but selecting a special projection of the processes that appears as time reversible. The final version of this approach leads to the concept of entwined paths [42]. It should be interesting to analyse the predictions of such an approach in the domain of statistical physics.

7. Concluding remarks

Starting from a dynamical point of view in which we give a physical meaning to the paths, we rederive the standard results of equilibrium thermodynamics as shown by (14). To describe the thermal equilibrium, the paths are investigated on a time $\tau = \beta\hbar$ which is related to an equilibrium condition. The entropy is defined in terms of order in a given spacetime. For ordinary systems, this definition and Boltzmann definition lead to the same value for the entropy. A relation between action and entropy has been established, it is identical to the one existing for black holes although the underlying physics is very different of the one existing for ordinary systems.

Recent results show that, possibly, the concept of entropy is far to be well understood [3, 43]. This is the case in the black hole domain in which we are not able to associate the entropy with a counting of microstates. In addition, as a consequence of the so-called Unruh effect [43], it appears that the number of microstates might be related to the motion of the observer and, as a consequence, the Boltzmann entropy should lose its fundamental character. This suggests that a definition of entropy more general that the one introduced by Boltzmann might exist. This paper is an attempt to introduce a new definition of entropy that we can use both for ordinary systems and systems including black hole. It is also an illustration showing that general results obtained in the domain of black hole can be extended to ordinary systems producing an improvement in our general understanding about thermodynamics.

References

[1] Hawking S W 1975 *Commun. Math. Phys.* **43** 199
[2] Bekenstein J D 1973 *Phys. Rev.* D **7** 2333
[3] Wald R M 1999 *Class. Quantum Grav.* **16** 177
See also Wald R M 1997 *Preprint* gr-qc/9702022
[4] Bekenstein J D 1996 *Proc. 7th Marcel Grossmann Meeting* ed R T Jantzen, G M Kaiser and R Ruffini (Singapore: World Scientific) (*Preprint* gr-qc/9409015)
[5] Sorkin R D 2005 *Preprint* hep-th/0504037
[6] Fulling S A and Ruijsenaars S N M 1987 *Phys. Rep.* **152** 135
[7] Gibbons G W and Hawking S W 1977 *Phys. Rev.* D **15** 2572
[8] Badiali J P 2005 *J. Phys. A: Math. Gen.* **38** 2835 (*Preprint* quant-ph/0409138)
[9] Broglie L de 1986 *Diverses questions de mécanique et de thermodynamique classiques et relativistes* (Berlin: Springer)
[10] Feynman R P and Hibbs A R 1965 *Quantum Mechanics and Path Integrals* (New York: Mc-Graw Hill) chapter 10
[11] Buniy R V, Hsu S D H and Zee A 2005 *Preprint* hep-th 0508039
[12] Snyder H S 1947 *Phys. Rev.* **71** 38
[13] Sidharth B G 2000 *Chaos Solitons Fractals* **11** 1269
[14] Garay L J 1995 *Int. J. Mod. Phys.* A **10** 145
[15] Doplicher S, Fredenhagen K and Roberts J E 1994 *Phys. Lett.* B **331** 39
[16] Bombelli L, Lee L, Mayer D and Sorkin R D 1987 *Phys. Rev. Lett.* **59** 521

[17] Rideout D P and Sorkin R D 1999 *Phys. Rev.* D **61** 024002
[18] Martin X, O'Connor D and Sorkin R 2005 *Phys. Rev.* D **71** 024029
[19] Itzykson C and Drouffe J M 1989 *Statistical Field Theory* (Cambridge: Cambridge University Press)
[20] Nagasawa M 2000 *Stochastic Processes in Quantum Physics (Monographs in Mathematics* vol 94*)* (Basel: Birkhauser)
[21] Badiali J P 1999 *Phys. Rev.* E **60** 2533
[22] Badiali J P 2003 *Condens. Matter Phys.* **6** 1
[23] Giles R 1964 *Mathematical Foundations of Thermodynamics (International Series of Monographs on Pure and Applied Mathematics)* (New York: McMillan)
[24] Hawking S and Penrose R 1996 *The Nature of Space and Time* (Princeton: Princeton University Press)
[25] Padmanabhan T 2002 *Class. Quantum Grav.* **19** 5387
[26] Le Bellac M and Barton G 1988 *Quantum and Statistical Field Theory* (Portland, OR: Booknews)
[27] Einstein A 1982 *Autobiographical Notes* ed P A Schilpp (La Salle, IL: Open court Publishing Company)
[28] Jacobson T 1195 *Phys. Rev. Lett.* **75** 1260 (*Preprint* gr-qc/9504004)
[29] Planck M 1945 *Treatise on Thermodynamics* (New York: Dover)
[30] Nagasawa M 1993 *Schrödinger Equation and Diffusion Theory* (Basel: Birkhauser)
[31] Grabert H, Hanggi P and Talkner P 1979 *Phys. Rev.* A **19** 2440
[32] Wang M S and Liang W-K 1993 *Phys. Rev.* D **48** 1875
[33] Badiali J P in preparation
[34] Kröger H 2000 *Phys. Rep.* **323** 81
[35] Nottale L 1992 *Fractal Space-Time and Microphysics, Towards a Theory of Scale Relativity* (Singapore: World Scientific)
[36] Ord G N 1983 *J. Phys. A: Gen. Phys.* **16** 1869
[37] Célérier M N and Nottale L 2004 *J. Phys. A: Math. Gen.* **37** 931
[38] El Naschie M S 1997 *Chaos Solitons Fractals* **9** 1211
[39] Nelson E 1966 *Phys. Rev.* **150** 1076
[40] Ord G N 1992 *Int. J. Theor. Phys.* **31** 1177
[41] McKeon D G C and Ord G N 1992 *Phys. Rev. Lett.* **69** 3
[42] Ord G N and Mann R B 2003 *Phys. Rev.* A **67** 022105
[43] Marolf D, Minic D and Simon S F 2004 *Phys. Rev.* D **69** 064004 (*Preprint* hep-th/0310022)